$$J_3(x) = 4x^{-1}\left(2x^{-1}J_1(x) - J_0(x)\right) - J_1(x)\Big|_1^2$$

$$J_3(x) = 8x^{-2}J_1(x) - 4x^{-1}J_0(x) - J_1(x)\Big|_1^2$$

$$= 8x^{-2}J_1(x) - 4x^{-1}J_0(x) - J_1(x) - \left(8x^{-2}J_1(x)\right.$$

$$\left.- 4x^{-1}J_0(x) - J_1(x)\right).$$

$$= 8x^{-2}J_1(2) - 4\cancel{x^{-1}J_0(2)} - \cancel{J_1(2)} - 8x^{-2}J_1(1)$$

$$+ 4\cancel{x^{-1}J_0(1)} + \cancel{J_1(1)}.$$

$$= 8\frac{1}{4}J_1(2) - \frac{4}{2}J_0(2) - J_1(2) - 8J_1(1)$$

$$+ 4J_0(1) + J_1(1).$$

$$= 8\frac{1}{4}J_1(2) - J_1(2) - 8J_1(1) + J_1(1) - 2J_0(2)$$

$$+ 4J_0(1)$$

General Solution - Linear Dependance :-

For general Solution of Bessel's equation (1), Replacing ν by $-\nu$ in (12-1), we have

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(m-\nu+1)}.$$

Since Bessel's equation involves ν^2, the function J_ν and $J_{-\nu}$ are solutions of the equation for the same ν. If ν is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x).$$ (Linear independent)

This cannot be the general Solution for an integer $\nu = n$ because in that case, we have linear independance. because for $\nu = n$, we have linear dependance because

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (n = 1, 2, \cdots)$$

Bessel function $Y_\nu(x)$. General Solution :-

To obtain a general solution of Bessel's equation (1) for any ν, we now introduce "Bessel functions of the second kind" $Y_\nu(x)$,

- when $\nu = n = 0$

the Bessel's equation can be written (divide by x).

$$x y'' + y' + x y = 0 \quad \text{------} (1)$$

now we have a double root $r=0$, now we have only one solution, $J_0(x)$,

the desired second solution must be of the form,

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} A_m x^m \quad (1^*)$$

$$y_2'(x) = J_0 \frac{1}{x} + J_0' \ln x + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2''(x) = -J_0 \frac{1}{x^2} + J_0' \frac{1}{x} + J_0' \frac{1}{x} + J_0'' \ln x + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

Substitute all derivative to eq(1), we obtain

$$x J_0'' \ln x + 2J_0' - \cancel{\frac{J_0}{x}} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-1} + J_0' \ln x + \cancel{\frac{J_0}{x}} + \sum_{m=1}^{\infty} m A_m x^{m-1} + x J_0(x) \ln x + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

because J_0 is a solution of equation (1) then $xJ'' \ln x$, $J_0' \ln x$ and $x J_0 \ln x$ is zero., we get

$$2J_0' + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-1} + \sum_{m=1}^{\infty} m A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

for the first and second series

$$\sum_{m=1}^{\infty} m(m-1) A_m x^{m-1} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$= \sum_{m=1}^{\infty} m^2 - m + m A_m x^{m-1}$$

$$= \sum_{m=1}^{\infty} m^2 A_m x^{m-1}$$

We have $J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2}$, and $\frac{m!}{m} = (m-1)!$

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2m\, x^{2m-1}}{2^{2m} (m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m! (m-1)!} \qquad (2)$$

Together with $\sum m^2 A_m x^{m-1}$ and $\sum A_m x^{m+1}$

$$\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0 \quad (3)$$

A_m with odd subscripts are all zero. Assume $m-1=2s$ in the second term series. The first term series doesn't have A, just in the second and third term series, the third series, $m+1=2s$

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0, \quad s=1,2,\ldots$$

Since $A_1=0, \Rightarrow A_3=0, A_5=0, \ldots$

equate the sum of the coefficients of x^{2s+1} to zero.

at $s=0 \Rightarrow -1+4A_2=0 \Rightarrow A_2=\frac{1}{4}$

in the first series in (3) $\Rightarrow 2m-1=2s+1$, Hence $m=s+1$, in the second $m-1=2s+1$, and in the third $m+1=2s+1$, obtain

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!\,s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$$

For $s=1 \Rightarrow \frac{1}{8}+16A_4+A_2=0$, thus $A_4=-\frac{3}{128}$

in general

$$A_{2m}=\frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right), \quad m=1,2,\ldots$$

~~Substitute~~

let $h_1=1$, $h_m=1+\frac{1}{2}+\cdots+\frac{1}{m}$, $m=2,3,\ldots$ ------ (4)

Substitute (4) in (1*), we obtain

$$y_2(x)=J_0(x)\ln x+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}$$

$$=J_0(x)\ln x+\frac{1}{4}x^2-\frac{3}{128}x^4+\frac{11}{13824}x^6-\cdots+\cdots$$

Since J_0 and y_2 are linearly independent functions, if we replace y_2 by an independent particular solution of the form $a(y_2+bJ_0)$, where $a\neq 0$, and b are constants. It is customary to choose $a=\frac{2}{\pi}$ and $b=\gamma-\ln 2$, where the number $\gamma=0.57721566490\ldots$ is the so called "Euler Constant" which is defined as the limit of

$$1+\frac{1}{2}+ \cdots +\frac{1}{s} - \ln s$$

The standard particular solution thus obtained is called the "Bessel function of the second kind" of order zero or "Neumann's function" of order zero and is denoted by $Y_0(x)$,

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2}+\gamma\right)+\sum_{m=}^{\infty}\frac{(-1)^{m-1} h_m x^{2m}}{2^{2m}(m!)^2}\right] \quad (5)$$

for small $x > 0$, the function $Y_0(x)$ behaves about like $\ln x$, and $Y_0(x) \longrightarrow -\infty$ as $x \longrightarrow 0$.

Bessel function of the second kind $Y_n(X)$.

For $\nu = n = 1, 2, \ldots$;

the standard second solution $Y_\nu(x)$ defined for all ν by the formula

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\left[J_\nu(x) \cos \nu\pi - J_{-\nu}(x)\right] \quad \text{(6-a)} \; \text{(a)}$$

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x). \quad \text{(6-b)} \; \text{(b)}$$

This function is called the "Bessel function of the second kind" of order ν or "Neumann's function" of order ν.

The series development of $Y_n(x)$ can be obtained if we insert the series (12-1) and eq (1*) for $J_\nu(x)$ and $J_{-\nu}(x)$ into (6-a) and then let ν approaches n,

$$Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + \gamma\right) + \frac{x^n}{\pi}\sum \frac{(-1)^{m-1}(h_m + h_{m-n})}{2^{2m+n}\, m!\,(m+n)!} * x^{2m}$$

$$- \frac{x^{-n}}{\pi}\sum_{m=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n}\, m!} x^{2m} \quad \text{(7)}$$

where $X > 0$, $n = 0, 1, \cdots$ and $h_0 = 0$, $h_1 = 1$,

$$h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \quad h_{m+n} = 1 + \frac{1}{2} + \cdots \frac{1}{m+n}$$

For $n = 0$, the last sum in (17) is to be replaced by 0, Furthermore, it can be shown that

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

$\therefore$ The general solution of Bessel's equation for all values of ν and $X > 0$ is

$$Y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

we finally mention that there is a particular need for solutions of "Bessel's equation" that are complex for values of X. For this purpose the solution

$$H_\nu(x) = J_\nu(x) + i\, Y_\nu(x)$$
$$H_\nu(x) = J_\nu(x) - i\, Y_\nu(x).$$

this called the "Bessel function of the third Kind of order ν or first and second "Hankel function" of order ν.

power series

Homework 1

Q1 Solve the differential equation by power series?

$$(1-x^2)y'' - 2xy' + 2y = 0$$

Q2 Solve the differential equation by Legendre equation.

1- $(1-x^2)y'' - 2xy' - 3y = 0$

2- $(1-x^2)y'' - 2xy' + 30y = 0$

3- $5(1-x^2)y'' - 10xy' + 150y = 0$

Q3 Find the series of

1- $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$

2- $\sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$

3- $\sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m x^{2m}$

Q4 Find the power series of

1- $y' = -2xy$

2- $y'' + y = 0$

3- $y'' - y' + xy = 0$

4- $(1-x^2)y'' - 2xy' + 2y = 0$

Power Series   Homework 2

Q1 Solve the differential equation by the Frobenius method.

(a) $(x^2 - x)y'' - xy' + y = 0$

(b) $(x+2)^2 y'' + (x+2)y' - y = 0$

(c) $xy'' + 2y' + xy = 0$

(d) $xy'' + y = 0$

(e) $xy'' + (2x+1)y' + (x+1)y = 0$

(f) $xy'' + (1-2x)y' + (x-1)y = 0$

(g) $xy'' + (2-2x)y' + (x-2)y = 0$

power Series Homework 3

Q Find a general solution in terms of J_ν and $J_{-\nu}$ for:-

a) $x^2y'' + xy' + (x^2 - \frac{4}{49})y = 0$

b) $y'' + y' + \frac{1}{4}y = 0$ $(\sqrt{x} = z)$

c) $x^2y'' + xy' + (x^2 - 16)y = 0$

d) $xy'' + y' + 36y = 0$ $(12\sqrt{x} = z)$